8 Beams on Elastic Foundation

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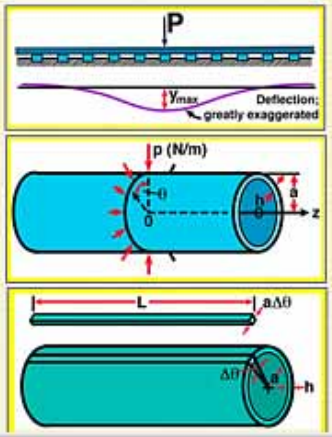
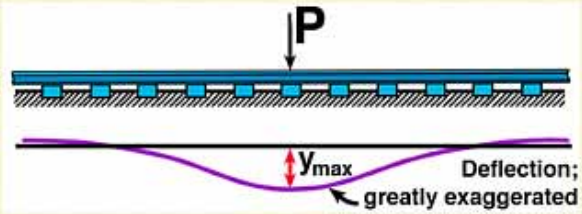
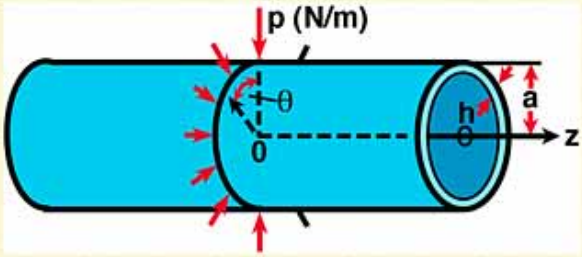
Examples

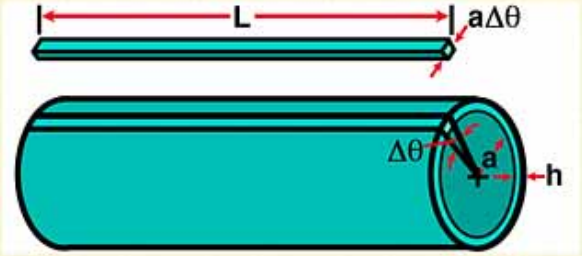
Steel railroad rail resting on timber cross ties. Elastic foundation consists of the ties, the ballast and the subgrade.

Thin-walled circular cylinder subjected to rotationally symmetrical loads.





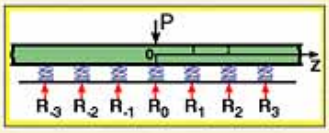


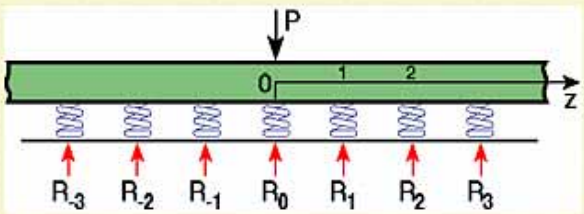


Types of Foundation

Continuous or discontinuous (discrete).

Linear or nonlinear.

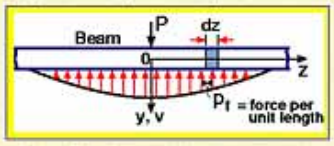




Beams on Winkler Foundation

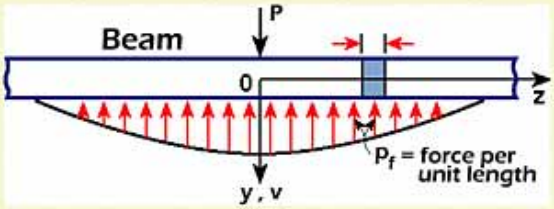
Linearly Elastic Foundation Models
Winkler Model (1867)

The intensity of the pressure (reaction) developed at any point is proportional to the deflection of the beam at that point. It is independent of the pressure at nearby joints.



Foundation consists of a series of independent springs (one-parameter foundation)

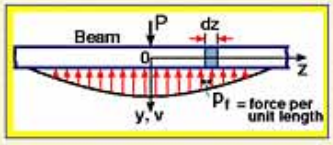
Deflection at every point of the foundation is proportional to the pressure applied at that point. It is independent of the pressures at nearby points.



Beams on Winkler Foundation

Linearly Elastic Foundation Models
Winkler Model (1867)

Deflection at every point of the foundation is proportional to the pressure applied at that point. It is independent of the pressures at nearby points.



$p_f = \kappa v$ where κ = foundation modulus – force/unit area

Beams on Winkler Foundation

Pasternak Model (1954)

Elastic layer - Winkler springs connected by a shear layer with stiffness K_1 (two-parameter foundation).

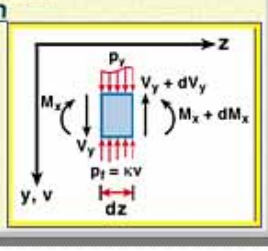
$$p_f = \kappa v - \kappa_1 \frac{d^2 v}{dz^2}$$

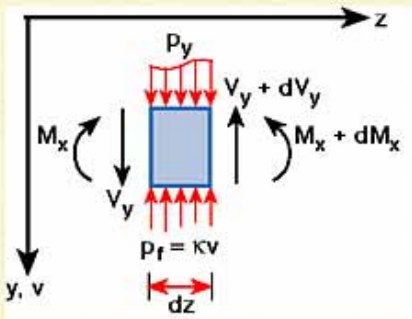
Governing Equations for a Straight Beam on a Winkler Foundation

Static Relations - Equilibrium

$$\frac{dV_y}{dz} = p_y - \kappa v$$

$$\frac{dM_x}{dz} = -V_y$$





Beams on Winkler Foundation

Governing Equations for a Straight Beam on a Winkler Foundation

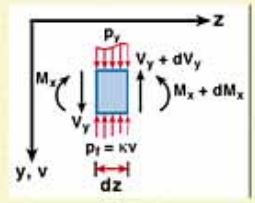
Static Relations - Equilibrium

$$\frac{dV_y}{dz} = p_y - \kappa v$$

$$\frac{dM_x}{dz} = -V_y$$

or

$$\frac{d^2M_x}{dz^2} = -p_y + \kappa v$$

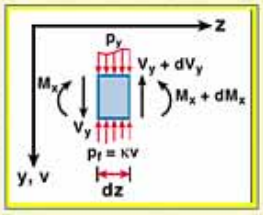


Beams on Winkler Foundation

Governing Equations for a Straight Beam on a Winkler Foundation

Kinematic Relation

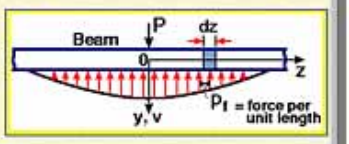
$$\kappa_x = -\frac{d^2v}{dz^2}$$

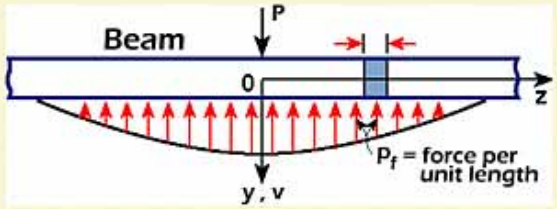


Constitutive Relation

$$M_x = EI_x \kappa_x$$

$$= -EI_x \frac{d^2v}{dz^2}$$



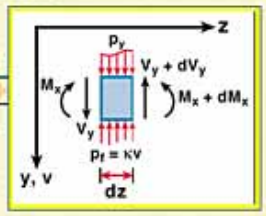


Beams on Winkler Foundation

Governing Equations for a Straight Beam on a Winkler Foundation

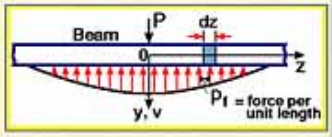
Governing Equation

$$\frac{d^2}{dz^2}\left(EI_x \frac{d^2 v}{dz^2}\right) = p_y - \kappa v$$



for uniform beams

$$EI_x \frac{d^4 v}{dz^4} + \kappa v = p_y$$



Beams on Winkler Foundation

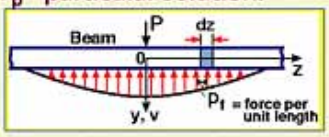
Governing Equations for a Straight Beam on a Winkler Foundation

General Solution

Let $\beta = \sqrt[4]{\frac{\kappa}{4\,EI_x}}$

$$v = e^{\beta z}(C_1 \cos \beta z + C_2 \sin \beta z) + e^{-\beta z}(C_3 \cos \beta z + C_4 \sin \beta z) + v_p$$

where v_p = particular solution.

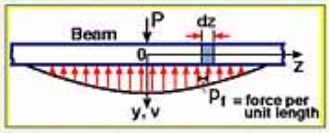


Beams on Winkler Foundation

Governing Equations for a Straight Beam on a Winkler Foundation

In terms of hyperbolic functions

$$v = [(C_1 + C_3)\cos\beta z + (C_2 + C_4)\sin\beta z]\cosh\beta z + [(C_1 + C_3)\cos\beta z + (C_2 + C_4)\sin\beta z]\sinh\beta z + v_p$$



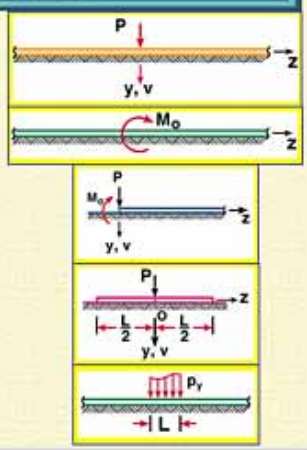
Special Cases

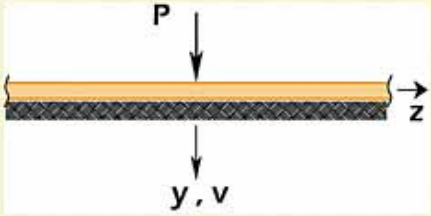
Infinite beams with concentrated loads

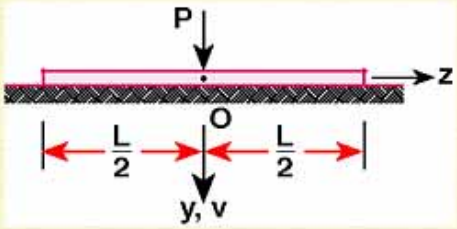
Semi-infinite beams with concentrated loads

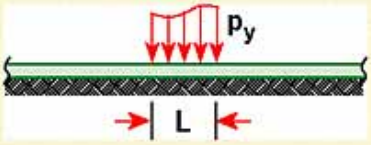
Beams of finite length

Distributed loads







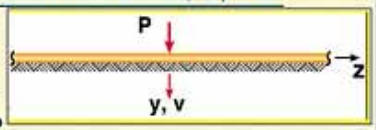


Special Cases

Infinite Beams with Concentrated Loads, $v_P = 0$

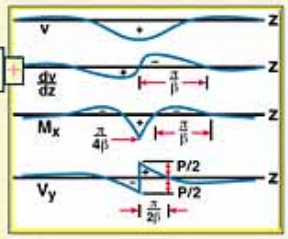
The origin is selected to be at the location of the concentrated load P

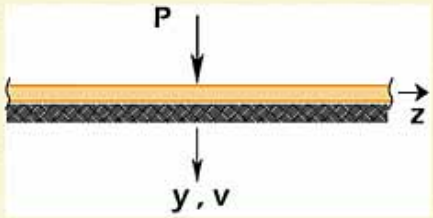
The deflection, slope, shear and moment are assumed to be zero at the ends.

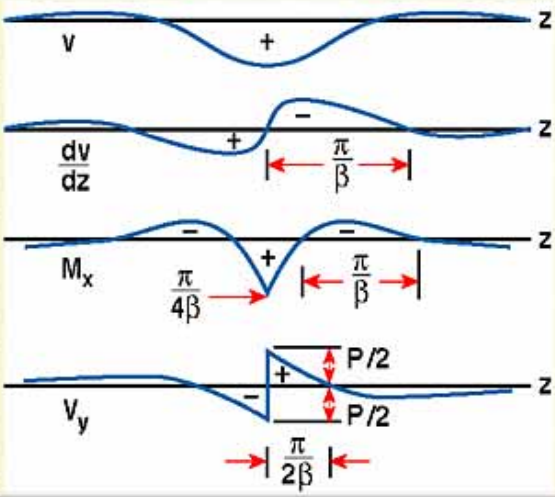


$$C_1 = C_2 = 0$$

$$v = e^{-\beta z}\left(C_3 \cos \beta z + C_4 \sin \beta z\right)$$

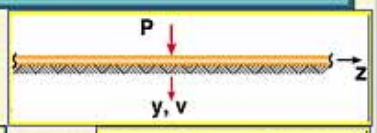






Special Cases

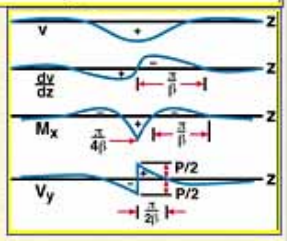
Conditions at $z = 0$



1. $$\frac{dv}{dz} = 0$$
$$-\beta(C_4 - C_3) = 0$$
$$\text{or, } C_4 = C_3 = C$$

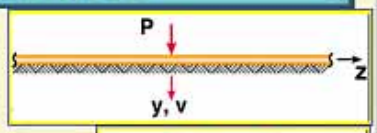
2. $$2\int_0^\infty \kappa v \, dz = P$$

$$V_y\big|_{z=0} = -EI\,\frac{d^3v}{dz^3}\bigg|_{z=0} = -\frac{P}{2}$$



Special Cases

Conditions at $z = 0$

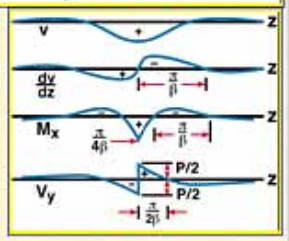


$$v_{max} = \frac{P\beta}{2\kappa}$$

$$M_x = \frac{P}{4\beta}$$

$$p_f = \frac{P\beta}{2}$$

$$\text{or, } C = \frac{P\beta}{2\kappa}$$



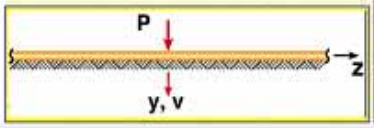
$$v = \frac{P\beta}{2\kappa} e^{-\beta z}\left(\cos \beta z + \sin \beta z\right)$$

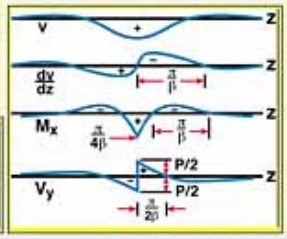
Special Cases

Conditions at $z = 0$

$$v = \frac{P\beta}{2\kappa} e^{-\beta z}\left(\cos \beta z + \sin \beta z\right)$$

$$\frac{dv}{dz} = -\frac{P\beta^2}{\kappa} e^{-\beta z} \sin \beta z$$

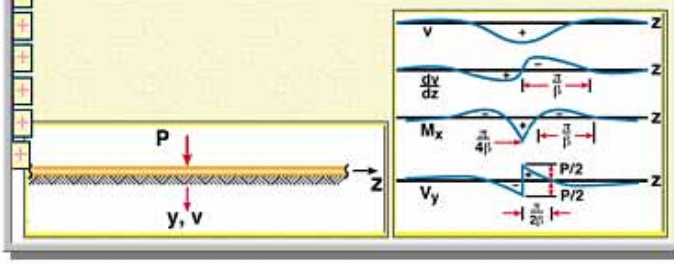




Special Cases

Conditions at $z = 0$

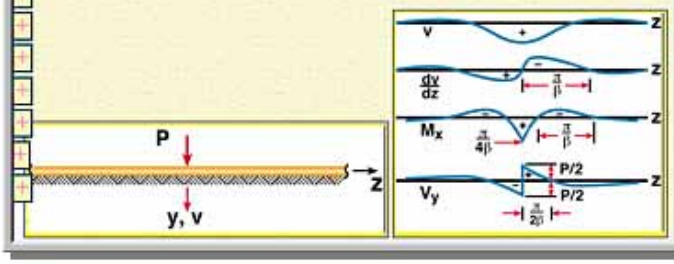
$$EI\frac{d^2v}{dz^2} = -M_x = -\frac{P}{4\beta}e^{-\beta z}(\cos\beta z - \sin\beta z)$$



Special Cases

Conditions at $z = 0$

$$EI\frac{d^3v}{dz^3} = V_y = \frac{P}{2}e^{-\beta z}(\cos\beta z)$$



Special Cases

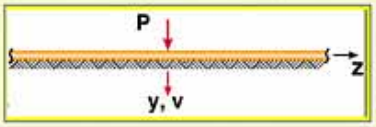
What does infinitely long beams mean?

The expression for the displacement of **v**

$$v = \frac{P\beta}{2\kappa}e^{-\beta z}(\cos\beta z + \sin\beta z)$$

can be written in the form

$$v = \frac{P\beta}{\sqrt{2}\,\kappa}e^{-\beta z}\sin\left(\beta z + \frac{\pi}{4}\right)$$

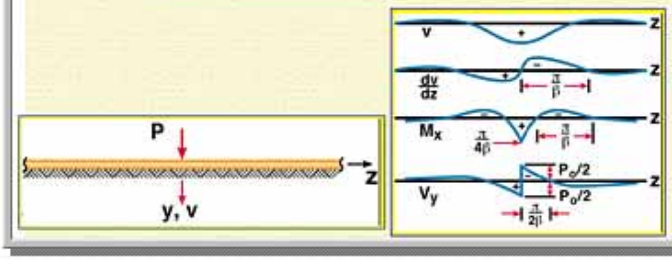


Special Cases

What does infinitely long beams mean?

- It is a damped sinusoid.
- The wavelength λ_w is given by

$$\lambda_w = \frac{2\pi}{\beta} = 2\pi\sqrt[4]{\frac{4\,EI_x}{\kappa}}$$

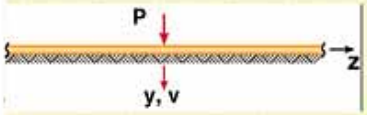


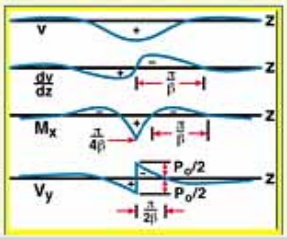
Special Cases

What does infinitely long beams mean?

If the wavelength λ_w is much smaller than the length of the beam, then the beam can be considered to be of infinite length.

$$\lambda_w = \frac{2\pi}{\beta} = 2\pi\sqrt[4]{\frac{4\,EI_x}{\kappa}}$$





Special Cases

Semi-Infinite Beam with Concentrated Loads at its End

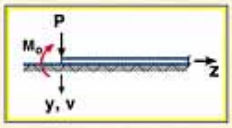
- The origin is selected to be at the loaded end.
- The deflection, slope, shear and moment are assumed to be zero at the far end.

$$C_1 = C_2 = 0 \qquad v = e^{-\beta z}(C_3\cos\beta z + C_4\sin\beta z)$$

Conditions at $z = 0$

1. $$M_0 = -EI\frac{d^2v}{dz^2}$$

2. $$P_0 = -EI\frac{d^3v}{dz^3}$$



Special Cases

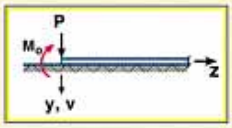
Semi-Infinite Beam with Concentrated Loads at its End

Conditions at $z = 0$

$$M_0 = -EI\frac{d^2v}{dz^2} \qquad P_0 = -EI\frac{d^3v}{dz^3}$$

$$\text{or, } C_3 = \frac{1}{\kappa}\left(2\beta^2 M_0\right)$$

$$C_4 = \frac{-2\beta}{\kappa}\left(P_0 + \beta M_0\right)$$



$$v = \frac{-2\beta}{\kappa} e^{-\beta z}\left[P_0 \cos\beta z + 2\beta M_0 (\cos\beta z - \sin\beta z)\right]$$

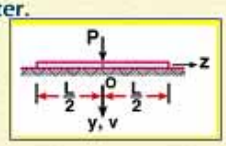
Special Cases

Beams of Finite Length Subjected to Concentrated Load at the Center

The origin is selected at the center.

Conditions at the center

$$\frac{dv}{dz} = 0 \quad , \quad V_y = \frac{P}{2}$$



Conditions at $z = \pm L/2$

$$M_x = 0$$

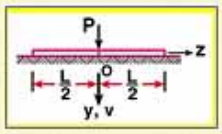
$$V_y = 0$$

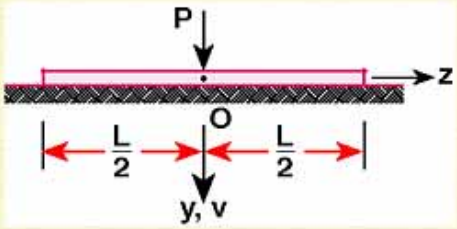
Special Cases

For a beam of finite length L subjected to a concentrated load at the center, if the origin is selected at the center

$$v_c = \frac{P\beta}{2\kappa}\,\frac{2 + \cos\beta L + \cosh\beta L}{\sin\beta L + \sinh\beta L}$$

$$M_c = \frac{P}{4\beta}\,\frac{\cosh\beta L + \cos\beta L}{\sin\beta L + \sinh\beta L}$$



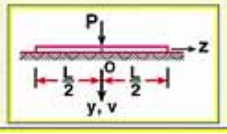


Special Cases

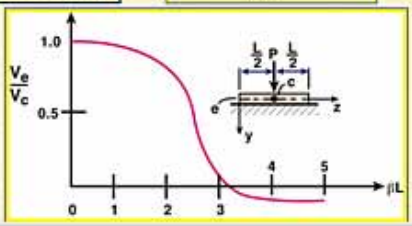
For a beam of finite length L subjected to a concentrated load at the center, if the origin is selected at the center

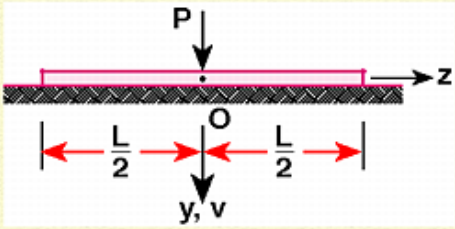
$$v_e = \frac{2P\beta}{\kappa}\,\frac{\cos(\beta L/2) + \cosh(\beta L/2)}{\sin\beta L + \sinh\beta L}$$

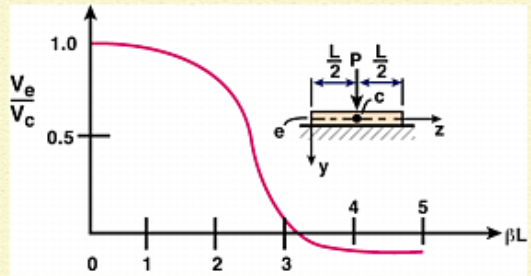
$$\frac{V_e}{V_c} = \frac{4\cos(\beta L/2)\cosh(\beta L/2)}{2 + \cos\beta L + \cosh\beta L}$$



$$\frac{L}{\lambda_w} = \frac{\beta L}{2\pi}$$





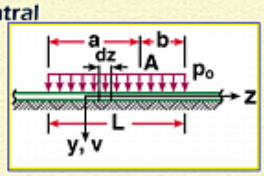


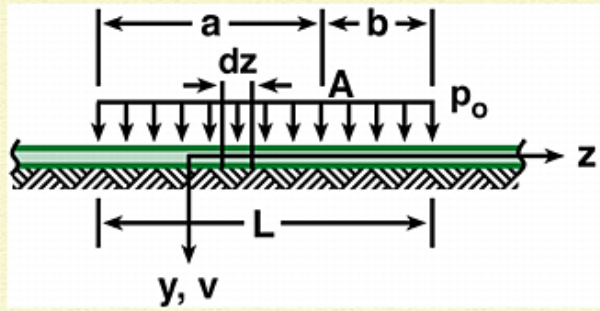
Special Cases

Infinite Beams Subjected to Distributed Loads

The loading is uniform over a central region, and the intensity is p_0

The loading on an element dz produces a displacement dv at point A





Special Cases

Infinite Beams Subjected to Distributed Loads

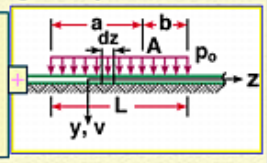
The loading is uniform over a central region, and the intensity is p_0

The loading on an element dz produces a displacement dv at point A

$$dv = p_o\, dz\, \frac{\beta}{2\kappa}\, e^{-\beta z}\,(\cos \beta z + \sin \beta z)$$

The total displacement at point A is given by

$$v_A = \frac{p_o \beta}{2\kappa}\int_0^a e^{-\beta z}(\cos \beta z + \sin \beta z)dz + \int_0^b e^{-\beta z}(\cos \beta z + \sin \beta z)dz$$

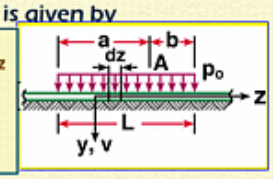


Special Cases

Infinite Beams Subjected to Distributed Loads

The total displacement at point A is given by

$$v_A = \frac{p_o \beta}{2\kappa}\int_0^a e^{-\beta z}(\cos \beta z + \sin \beta z)dz + \int_0^b e^{-\beta z}(\cos \beta z + \sin \beta z)dz$$



where a and b are the distances from point A to the extremities of the loaded area

or,

$$v = \frac{p_o}{2\kappa}\left[2 - e^{-\beta a}\cos \beta a - e^{-\beta b}\cos \beta b\right]$$

The total bending moment is given by

$$M = \frac{p_o}{4\beta^2}\left[e^{-\beta a}\sin \beta a + e^{-\beta b}\sin \beta b\right]$$

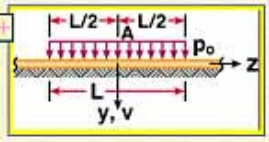
Special Cases

Infinite Beams Subjected to Distributed Loads

At the midpoint of the loaded area, a = b = L/2

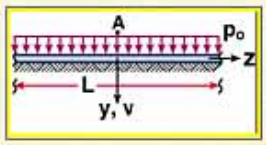
$$v_o = \frac{p_o}{\kappa}\left[1 - e^{-\beta L/2}\cos\frac{\beta L}{2}\right]$$

$$M_0 = \frac{p_o}{2\beta^2}e^{-\beta L/2}\sin\frac{\beta L}{2}$$



If L is large, the above formulas reduce to

$$v_o = \frac{p_o}{\kappa}, \quad M_0 = \frac{p_o}{2\beta^2}$$



Special Cases

Infinite Beams Subjected to Distributed Loads

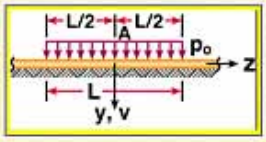
Define the four functions:

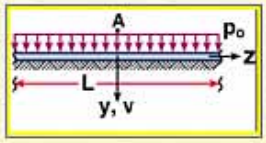
$$A_{\beta z} = e^{-\beta z}(\cos\beta z + \sin\beta z)$$

$$B_{\beta z} = e^{-\beta z}\sin\beta z$$

$$C_{\beta z} = e^{-\beta z}(\cos\beta z - \sin\beta z)$$

$$D_{\beta z} = e^{-\beta z}\cos\beta z$$





Special Cases

Infinite Beams Subjected to Distributed Loads

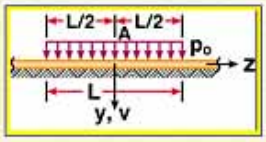
Define the four functions:

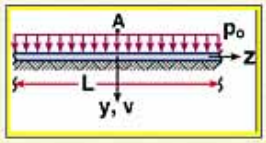
$$v = \frac{P\beta}{2\kappa}A_{\beta z}$$

$$\frac{dv}{dz} = -\frac{P\beta^2}{\kappa}B_{\beta z}$$

$$M_x = \frac{P}{4\beta}C_{\beta z}$$

$$V_y = \frac{P}{2}D_{\beta z}$$





Special Cases

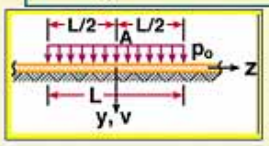
Infinite Beams Subjected to Distributed Loads

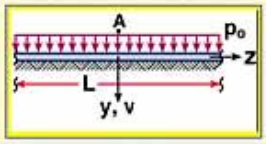
Define the four functions:

$$= \frac{p_o\beta}{2\kappa}\left[\int_0^a A_{\beta z}\,dz + \int_0^b A_{\beta z}\,dz\right]$$

$$= \frac{p_o}{2\kappa}\left[2 - D_{\beta a} - D_{\beta b}\right]$$

$$M_A = \frac{p_o}{4\beta^2}\left[B_{\beta a} - B_{\beta b}\right]$$





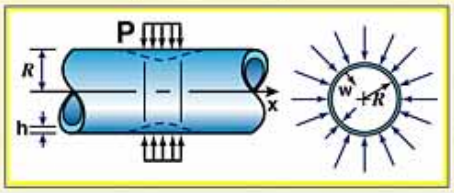
Thin-walled Circular Cylinder Subjected to Axially Symmetric Loads

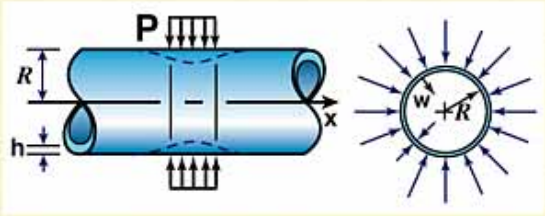
Governing Differential Equation

$$D\frac{d^4w}{dx^4} + \frac{Eh}{R^2}w = p$$

where

$$D = \frac{Eh^3}{12(1-\upsilon^2)}$$

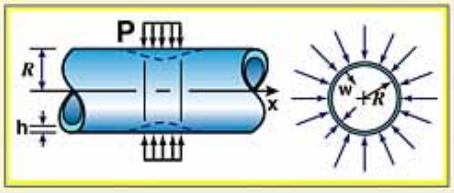




Thin-walled Circular Cylinder Subjected to Axially Symmetric Loads

The equation is of the same form as that for beam on elastic foundation

$w \leftrightarrow v$ $\quad D \leftrightarrow EI_x$ $\quad \frac{Eh}{R^2} \leftrightarrow \kappa$



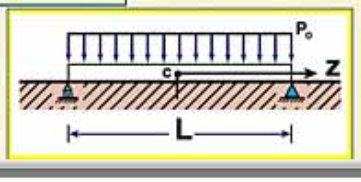
Examples

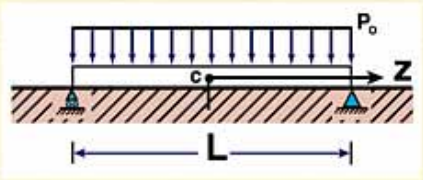
Simply supported beam of finite length on Winkler Foundation, subjected to uniformly distributed loading

$$v = (A_1 \cos \beta z + A_2 \sin \beta z)\cosh \beta z + (A_3 \cos \beta z + A_4 \sin \beta z)\sinh \beta z$$

Choosing the origin to be at the center, then

$A_2 = A_3 = 0$ due to symmetry





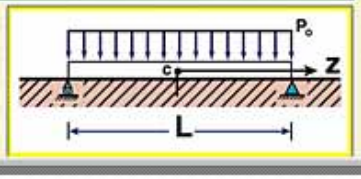
Examples

from which

$$A_1 = \frac{-p_o}{\kappa}\,\frac{2\cos(\beta L/2)\cosh(\beta L/2)}{\cos \beta L + \cosh \beta L}$$

$$A_4 = \frac{-p_o}{\kappa}\,\frac{2\sin(\beta L/2)\sinh(\beta L/2)}{\cos \beta L + \cosh \beta L}$$

$$v_c = \frac{p_o}{\kappa}\left[1 - \frac{2\cos(\beta L/2)\cosh(\beta L/2)}{\cos \beta L + \cosh \beta L}\right]$$



Examples

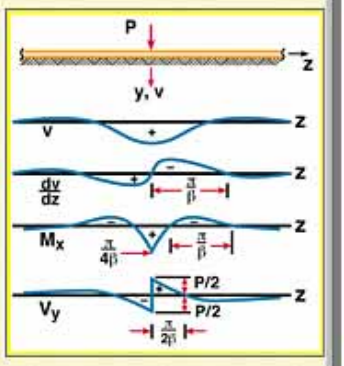
Symmetry Conditions

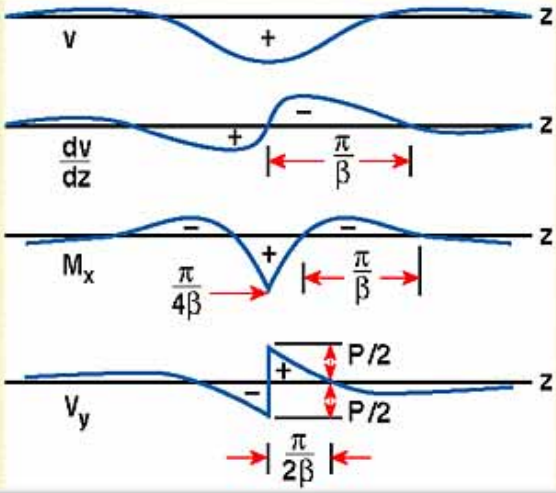
$$v(z) = v(-z)$$

$$\frac{dv}{dz}(z) = -\frac{dv}{dz}(-z)$$

$$\frac{d^2v}{dz^2}(z) = -\frac{d^2v}{dz^2}(-z)$$

$$\frac{d^3v}{dz^3}(z) = -\frac{d^3v}{dz^3}(-z)$$





Examples

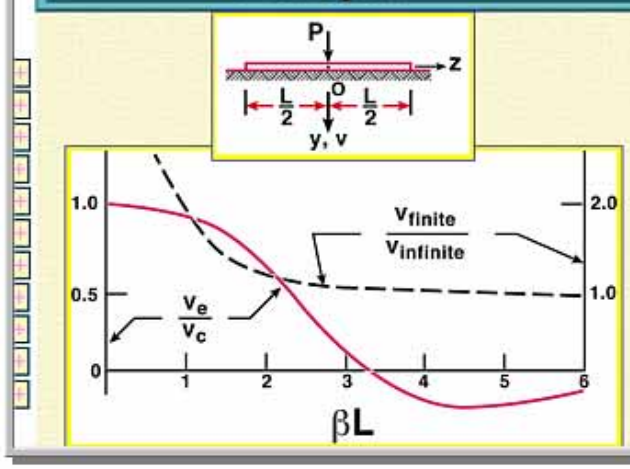
Symmetric (Even) Functions

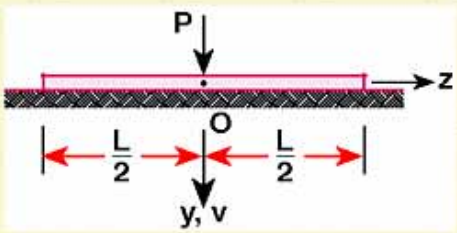
$\cos \beta z$, $\cosh \beta z$, $\cos \beta z \cosh \beta z$, $\sin \beta z \sinh \beta z$

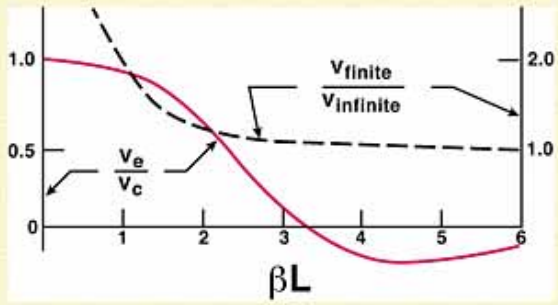
Antisymmetric (Odd) Functions

$\sin \beta z$, $\sinh \beta z$, $\sin \beta z \cosh \beta z$, $\cos \beta z \sinh \beta z$

Examples







Examples

β_z	$A_{\beta z}$	$B_{\beta z}$	$C_{\beta z}$	$D_{\beta z}$
0	1	0	1	1
0.001	1.0000	0.0010	0.9980	0.9990
0.002	1.0000	0.0020	0.9960	0.9980
0.003	1.0000	0.0030	0.9940	0.9970
0.004	1.0000	0.0040	0.9920	0.9960
0.005	1.0000	0.0050	0.9900	0.9950
0.006	1.0000	0.0060	0.9880	0.9940
0.007	0.9999	0.0070	0.9861	0.9930
0.008	0.9999	0.0080	0.9841	0.9920
0.009	0.9999	0.0087	0.9821	0.9910
0.010	0.9999	0.0099	0.9801	0.9900
0.011	0.9999	0.0109	0.9781	0.9890
0.012	0.9999	0.0119	0.9761	0.9880
0.013	0.9998	0.0129	0.9742	0.9870
0.014	0.9998	0.0138	0.9722	0.9860
0.015	0.9998	0.0148	0.9702	0.9850
0.016	0.9997	0.0158	0.9683	0.9840
0.017	0.9997	0.0167	0.9663	0.9830
0.018	0.9997	0.0177	0.9643	0.9820
0.019	0.9996	0.0187	0.9624	0.9810
0.02	0.9996	0.0196	0.9604	0.9800
0.03	0.9991	0.0291	0.9409	0.9700
0.04	0.9984	0.0384	0.9216	0.9600
0.05	0.9976	0.0476	0.9025	0.9501

Examples

β_z	$A_{\beta z}$	$B_{\beta z}$	$C_{\beta z}$	$D_{\beta z}$
0.10	0.9906	0.0903	0.8100	0.9003
0.15	0.9796	0.1283	0.7224	0.8510
0.20	0.9651	0.1627	0.6398	0.8024
0.25	0.9472	0.1927	0.5619	0.7546
0.30	0.9267	0.2189	0.4888	0.7078
0.35	0.9036	0.2416	0.4204	0.6620
0.40	0.8784	0.2610	0.3564	0.6174
0.45	0.8513	0.2774	0.2968	0.5742
0.50	0.8231	0.2908	0.2414	0.5323
0.55	0.7934	0.3016	0.1902	0.4918
0.60	0.7628	0.3099	0.1430	0.4529
0.65	0.7313	0.3160	0.0996	0.4156
0.70	0.6997	0.3199	0.0599	0.3798
0.75	0.6676	0.3220	0.0237	0.3456
¼ π	0.6448	0.3224	0	0.3224
0.80	0.6353	0.3223	−0.0093	0.3131
0.85	0.6032	0.3212	−0.0391	0.2821
0.90	0.5712	0.3183	−0.0658	0.2527
0.95	0.5396	0.3146	−0.0896	0.2250
1.00	0.5083	0.3096	−0.1109	0.1987

Examples

β_z	$A_{\beta z}$	$B_{\beta z}$	$C_{\beta z}$	$D_{\beta z}$
3.15	−0.0432	−0.0004	−0.0424	−0.0428
3.20	−0.0431	−0.0024	−0.0383	−0.0407
3.25	−0.0427	−0.0042	−0.0343	−0.0385
3.30	−0.0422	−0.0058	−0.0306	−0.0365
3.35	−0.0417	−0.0073	−0.0271	−0.0344
3.40	−0.0408	−0.0085	−0.0238	−0.0323
3.45	−0.0399	−0.0097	−0.0206	−0.0303
3.50	−0.0388	−0.0106	−0.0177	−0.0283
3.55	−0.0378	−0.0114	−0.0149	−0.0264
3.60	−0.0366	−0.0121	−0.0124	−0.0245
3.65	−0.0354	−0.0126	−0.0101	−0.0227
3.70	−0.0341	−0.0131	−0.0079	−0.0210
3.75	−0.0327	−0.0134	−0.0059	−0.0193
3.80	−0.0314	−0.0137	−0.0040	−0.0177
3.85	−0.0300	−0.0139	−0.0023	−0.0162
3.90	−0.0286	−0.0140	−0.0008	−0.0147
5/4π	−0.0278	−0.0140	0	−0.0139
3.95	−0.0272	−0.0139	0.0005	−0.0133
4.00	−0.0258	−0.0139	0.0019	−0.0120
4.50	−0.0132	−0.0108	0.0085	−0.0023
3/2π	−0.0090	−0.0090	0.0090	0

Examples

β_z	$A_{\beta z}$	$B_{\beta z}$	$C_{\beta z}$	$D_{\beta z}$
3/2π	−0.0090	−0.0090	0.0090	0
5.00	−0.0046	−0.0065	0.0084	0.0019
7/4π	0	−0.0029	0.0058	0.0029
5.50	0.0000	−0.0029	0.0058	0.0029
6.00	0.0017	−0.0007	0.0031	0.0024
2π	0.0019	0	0.0019	0.0019
6.50	0.0018	0.0003	0.0012	0.0018
7.00	0.0013	0.0006	0.0001	0.0007
9/4π	0.0012	0.0006	0	0.0006
7.50	0.0007	0.0005	−0.0003	0.0002
5/2π	0.0004	0.0004	−0.0004	0