Solutions for some of Assignment 1 Problems

6. a) $\delta_{ij}\delta_{ik} = \delta_{1j}\delta_{1k} + \delta_{2j}\delta_{2k} + \delta_{3j}\delta_{3k}$

j = 1 : $\underline{\delta_{11}\delta_{1k}} + \delta_{21}\delta_{2k} + \delta_{31}\delta_{3k}$

only nonzero if k = 1

j = 2 : $\delta_{12}\delta_{1k} + \underline{\delta_{22}\delta_{2k}} + \delta_{32}\delta_{3k}$

only nonzero if k = 2

j = 3 : $\delta_{13}\delta_{1k} + \delta_{23}\delta_{2k} + \underline{\delta_{33}\delta_{3k}}$

only nonzero if k = 3

or $\delta_{ij}\delta_{ik} = \delta_{jk}$

Problem 7

$e_{pqs}\ e_{mnr}$

- Each of the six indices p, q, s, m, n, r can take only the values 1, 2, 3

- The nonzero e's correspond to three different indices (p, q, s) and (m, n, r)

- Therefore, each of the (p, q, s) should be equal to one of the (m, n, r). For nonzero terms, the other two indices of one set, must be related to the two indices of the other set and the result on whether the indices of one set form a cyclic or anticyclic permutation of the other set.

d) $e_{ijk} a_j a_k$

Since j and k are dummy indices, they can be interchanged.

$$e_{ijk} a_j a_k = e_{ikj} a_k a_j$$

But $a_k a_j = a_j a_k$

and $e_{ikj} = -e_{ijk}$

Therefore,

$$e_{ijk} a_j a_k = -e_{ijk} a_j a_k = 0$$

$$
\begin{array}{lll}
m = p & n = q \;,\; r = s & +1 \\
 & n = s \;,\; r = q & -1 \\
m = q & n = s \;,\; r = p & +1 \\
 & n = p \;,\; r = s & -1 \\
m = s & n = p \;,\; r = q & +1 \\
 & n = q \;,\; r = p & -1
\end{array}
$$

$$
\begin{aligned}
e_{pqs}\, e_{mnr} = {} & \delta_{mp}\left(\delta_{nq}\,\delta_{rs} - \delta_{ns}\,\delta_{rq}\right) \\
& + \delta_{mq}\left(\delta_{ns}\,\delta_{rp} - \delta_{np}\,\delta_{rs}\right) \\
& + \delta_{ms}\left(\delta_{np}\,\delta_{rq} - \delta_{nq}\,\delta_{rp}\right)
\end{aligned}
$$

Examples

Case	p	q	s	e_{pqs}	m	n	r	e_{mnr}	$e_{pqs}\, e_{mnr} =$
1	1	3	2	-1	2	1	3	-1	$\text{Det.}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1$
2	3	1	2	+1	1	3	2	-1	$\text{Det.}\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$

$$\text{Det} = \left|\begin{array}{ccc|cc} \delta_{mp} & \delta_{mq} & \delta_{ms} & \delta_{mp} & \delta_{mq} \\ \delta_{np} & \delta_{nq} & \delta_{ns} & \delta_{np} & \delta_{nq} \\ \delta_{rp} & \delta_{rq} & \delta_{rs} & \delta_{rp} & \delta_{rq} \end{array}\right|$$

$$\mathbf{Det} = \left| \begin{array}{ccc|cc} \delta_{mp} & \delta_{mq} & \delta_{ms} & \delta_{mp} & \delta_{mq} \\ \delta_{np} & \delta_{nq} & \delta_{ns} & \delta_{np} & \delta_{nq} \\ \delta_{rp} & \delta_{rq} & \delta_{rs} & \delta_{rp} & \delta_{rq} \end{array} \right.$$

$$\begin{aligned} e_{pqs}\, e_{mnr} = {} & \delta_{mp} \left(\delta_{nq}\, \delta_{rs} - \delta_{ns}\, \delta_{rq} \right) \\ & + \delta_{mq} \left(\delta_{ns}\, \delta_{rp} - \delta_{np}\, \delta_{rs} \right) \\ & + \delta_{ms} \left(\delta_{np}\, \delta_{rq} - \delta_{nq}\, \delta_{rp} \right) \end{aligned}$$

8. a) $\vec{\nabla} = \vec{\mathbf{i}}_i \, \partial_i$

$$= \vec{\mathbf{i}}_j \, \partial_j$$

$$\vec{\nabla} \times \vec{\nabla}\phi = \vec{\mathbf{i}}_i \, \partial_i \times \vec{\mathbf{i}}_j \, \partial_j \, \phi$$

$$= \mathbf{e}_{ijk} \, \partial_i \, \partial_j \, \phi \, \vec{\mathbf{i}}_k$$

$$= \mathbf{e}_{jik} \, \partial_j \, \partial_i \, \phi \, \vec{\mathbf{i}}_k$$

$$= -\mathbf{e}_{ijk} \, \partial_i \, \partial_j \, \phi \, \vec{\mathbf{i}}_k$$

$$= 0$$

9. Principal Values and Principal Directions

$$\mathbf{T}_{ij} = \begin{bmatrix} 7 & 3 & 0 \\ 3 & 7 & 4 \\ 0 & 4 & 7 \end{bmatrix} \rightarrow \mathbf{Det} \begin{vmatrix} 7-\lambda & 3 & 0 \\ 3 & 7-\lambda & 4 \\ 0 & 4 & 7-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 21\,\lambda^2 + 122\,\lambda - 168 = 0$$

$$\mathbf{I} = \mathbf{T}_{ii} = 21$$

$$\mathbf{II} = \frac{1}{2}\left(\mathbf{T}_{ii}\mathbf{T}_{jj} - \mathbf{T}_{ij}\mathbf{T}_{ji}\right) = 122$$

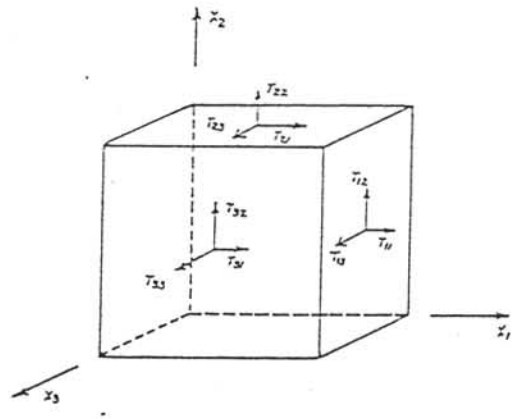
$$\mathbf{III} = \mathbf{det}\left|\mathbf{T}_{ij}\right| = 168$$

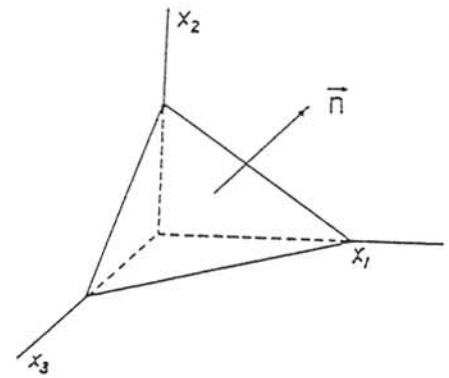
$$\lambda^{(1)} = 2\,, \quad \lambda^{(2)} = 7\,, \quad \lambda^{(3)} = 12$$

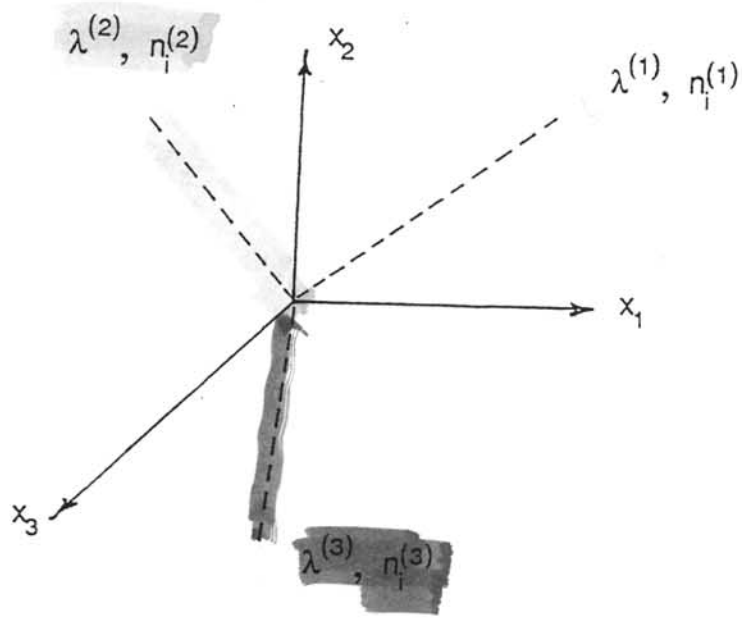
$$\begin{pmatrix} -\frac{3}{5\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ -\frac{2\sqrt{2}}{5} \end{pmatrix}, \quad \begin{pmatrix} \frac{4}{5} \\ 0 \\ -\frac{3}{5} \end{pmatrix}, \quad \begin{pmatrix} \frac{3}{5\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{2\sqrt{2}}{5} \end{pmatrix}$$

In the coordinate system represented by the principal directions

$$T_{i'j'} = \begin{bmatrix} \lambda^{(1)} & \cdot & \cdot \\ \cdot & \lambda^{(2)} & \cdot \\ \cdot & \cdot & \lambda^{(3)} \end{bmatrix}$$







Example.

$$\det = \begin{bmatrix} -\lambda & \mathbf{b}_{12} & \mathbf{b}_{13} \\ -\mathbf{b}_{12} & -\lambda & \mathbf{b}_{23} \\ -\mathbf{b}_{13} & -\mathbf{b}_{23} & -\lambda \end{bmatrix} = 0$$

or $$-\lambda\left[\lambda^2 + \mathbf{b}_{12}^2 + \mathbf{b}_{13}^2 + \mathbf{b}_{23}^2\right] = 0$$

The only real root is $\lambda = 0$.